Speed Of Trade And Arbitrage

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Abstract: We employ a theoretical microstructue model with overconfident traders (Kyle, Obizhaeva, Wang 2017) to demonstrate how market differences effect an arbitrageurs ability/willingness to engage in price correcting trades. The markets differ in the rates at which traders adjust their inventories towards their target levels, "trading speeds". The extent to which trading speed differences across markets effect the degree of price convergence or divergence depends on the arbitrageur's preexisting positions in each market. If the arbitrageur trades towards their target in each market at the same prevailing speed the prices converge. Trading speeds are found to be time invariant and the effect of a financial constraint on the arbitrageur increases in the difference between market speeds.

1 Introduction

This paper contemplates how differences in markets effects an arbitrageurs ability or willingness to engage in price correcting trading, enforcing the law of one price.

Theoretically if we take some arbitrary return r multiply it by the stochastic discount factor m and take the expectation of the product we should recover the fundamental pricing equation $1 = \langle mr \rangle$ (Cochrane 2009). Any asset pricing model, $H : returns \rightarrow prices$, amounts to an estimation of the homomorphism $\langle m \cdot \rangle$ which maps elements of a return space to some price space with any $r \notin ker(H)$ being seen as a mispricing in the eyes of the model. Since prices effect returns, asset pricing models amount to restrictions on the price generating process. In reality prices are the results of trading games (i.e. markets) which can be seen as a homomorphism $G : information \rightarrow prices$ that maps information to the price space. The assumptions made by a pricing model must necessarily restrict or prescribe the behavior of the participants of these games (H restricts G). One such assumption, which serves as a bedrock assumption underlying many asset pricing models, is the law of one price which is often justified through no-arbitrage arguments.

As financial economists studying prices we are often able to sidestep the dirty issue of individual preferences and whether or not they are rational by employing no-arbitrage arguments in order to evoke the law of one price. The general no-arbitrage argument is as follows: deviations from the law of one price are mispricings which present profit opportunities to arbitrageurs, taking advantage of these mispricings simultaneously serve to correct the mispricing, hence in equilibrium we can expect arbitrageurs to correct any mispricing. This argument allows us to remain agnostic concerning the origins of the mispricing and is a much less restrictive and presumptuous way of getting to the law of one price as opposed to getting it by making assumptions directly on the preferences of the underlying agents. The no-arbitrage argument does not hold perfectly in reality, a fact which has motivated the limits to arbitrage literature where apparent mispricings are explained by constraints on the arbitraguer's actions.

Returning to the sentiment expressed in the opening paragraph, we can think of

the arbitrageurs engaging in the price correcting trade as the behavior prescribed by the law of one price assumption (i.e. the restriction on the price generating process). Since the arbitrageur often takes positions in different markets, the law of one price "connects" the various trading games through the arbitrageur in order to justify itself. Intuitively constraining the arbitrageurs would weaken the law of one price. Even if we presume that the varying trading games are structurally the same in the sense that the games are played by the same rules, they can still vary in some underlying parameters, such as the number of players, their wealth levels, the precision of their signals and so on. So a natural question would be how differences in these parameters across the games effect, if at they do at all, the degree to which the arbitrageur engage in price correcting behavior. In this paper we model these kinds of differences by varying the number of traders in the markets by applying the smooth trading with overconfidence model which we borrow from Kyle, Obizhaeva, Wang (2017). The difference in market populations result in differences in trading speeds, the rate at which traders adjust their current inventories towards their target levels, and examine how these different trading speeds effect the degree to which an arbitrageur trades to correct a mispricing.

1.1 Literature Review

The question of how mispricings can exists in a world with arbitrageurs is one which is largely tackled by the limits to arbitrage literature, in particular this paper looks to contribute to the strand of the literature stemming from Shleifer & Vishny (1997) who use agency costs to constrain arbitrageurs by having investors pull their funds in the arbitrageur if the arbitrageur has too many losses, even though the arbitrage strategy's expected return increases.

Besides Kyle, Obizhaeva, & Wang (2017) which I build the paper around we will largely rely on the work done by Gromb & Vayanos (2002) who explicitly model the arbitrage opportunity by allowing arbitrageurs exclusive access to otherwise segmented markets and constraining them through the use of margin, making their constraint a function of the arbitrageurs wealth. Brunnermeier & Pedersen (2008) link market liquidity to funding

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liquidity, the topic of arbitrageur contagion as a wealth effect is covered by Kyle & Xiong (2001) where convergence traders can be forced to trade in direction of noise trade and Boyson, Stahel & Stulz (2010) who empirically find that the deterioration of hedge fund liquidity increases probability of contagion. As previously mentioned the use of the notion of "trading speed" relates directly to the work of Kyle & Obizhaeva (2016,2017) and their market microstructure invariance papers.

1.2   Model Overview

We employ a segmented market setup that is similar in spirit to that of Gromb & Vayanos (2002) where a single arbitrageur has unique access to two markets that trade the same underlying and trades to take advantage of their unique position. We differ by explicitly modeling the microstructure in each market using the model with overconfident traders from Kyle, Obizhaeva, Wang (2017), though we drop the inclusion of a public signal for simplicity.

General Structure/Assumptions   In the paper we explore three different specifications of our trading model, which can be roughly summarized by the following setup.

- There are two markets that are physically and informationally segmented but for a single arbitrageur .

- Each market $J \in \{A, Z\}$ has N_J number of non-arbitrageur traders (j-traders) plus the arbitrageur.

- Each j-trader observes a private signal of assets value, the arbitrageur does not observe a private signal.

- The j-traders are overconfident in their signal precision, and they all agree to disagree on it.

- The j-traders treat the arbitrageur as just another overconfident trader.

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- The arbitrageur knows that the assets will have the same liquidation value.

In the model, the markets effectively treat the arbitrageur as just another overconfident trader trades towards her target inventory at the equilibrium trading speed after observing some private signal. In actuality however the arbitrageur does not observe a private signal and trades towards her target at a different rate, where her trades and target inventories are functions of the price gap between the two markets. Whereas the trades and target inventories of the others are functions of the private signals they actually observed. If she instead were to trade towards her target inventory in each market at the prevailing trading speed then her resulting trades would have the effect that the prices would converge. In order to isolate the pure convergence trade from the portion which relied on the arbitrageurs' own (possibly incorrect) estimation of the liquidation value we introduced the no-information presumption which states that the arbitrageur views the prices in each market as containing no information at all about the liquidation value. The no-information presumption leaves the arbitrageur agnostic about the liquidation value and results in trades which solely rely on the arbitrageur's privileged position. The no-information presumption is analogous to an arbitrageur trading to profit on a price difference with no regard as to which price is more "correct" or what caused the initial divergence. An arbitrageur does not need know anything about cryptography to buy a bitcoin at $80 in one market and sell another at $100. Regardless of whether or not the arbitrageur presumes no-information the price difference in equilibrium would be halved if she had no initial inventory. However if she does have preexisting positions in the markets then under certain circumstances the price convergence could be less than expected or could even diverge. In a multiperiod setup it's assumed that the arbitrageur begins with no inventories and builds them up by trading over time, this allows for the situation in which the arbitrageur can trade in the direction of price divergence.

The rest of the paper is as follow: section 2 presents and solves a one period model, section 3 presents a multiperiod generalization of the single period model and demonstrates the invariance of trading speeds across time, section 4 considers a special case of a constrained arbitrageur in a two period version of the model and section 5 proposes a possible empirical test of the model and section 6 concludes.

2 One-Period Model

In this section we examine a single period segmented market model in which the market's traders observe a private signal concerning the tradeable's liquidation value and engage in one round of trading before the liquidation value is made public. We begin with two risky assets $J \in \{A, Z\}$, which are traded in separate markets, each in zero net supply, and three kinds of traders; A-traders, Z-traders, and an arbitrageur. A-traders exchange in asset A in market A and do not participate in market Z where asset Z is exchanged. Z-traders are similarly restricted to market Z where they exclusively trade in asset Z. Both A and Z traders are strategic in that they take into account the effect of their trades on the price and the trades of others. Were it not for the arbitrageur, the two markets would otherwise be segmented, allowing L_J to denote the population of participants in market $J \in \{A, Z\}$, the arbitrageur's "privileged position" is characterized as:

$$L_A = \{N_A, arb\}, \ L_Z = \{N_Z, arb\}, \ N_A \cap N_Z = \emptyset, \ L_A \cap L_Z = \{arb\} \tag{1}$$

where N_J denotes both the cardinality and the set of J-traders (i.e. $N_J = \{1, 2, ..., N_J\}$). We often refer to the arbitrary non-arbitrageur trade in market J as j or "j-trader".

An additional assumption to that of market population segmentation is that of informational segmentation which in this context simply means that any A-trader does not condition their trading decision s on the prevailing market price in Z and Z-traders do not condition on what's going on in market A. This lack of cross market can be interpreted in a number of ways, traders may not believe in or be ignorant of any relationship between assets A and Z, they may have limited attention or perhaps monitoring costs are too high so it may be too costly to monitor multiple possibly unrelated markets in order to trade in only one market.

Meanwhile the arbitrageur is aware of a relationship between the two assets, namely that the two assets will share the same liquidation value $v_A = v_Z = v$, knowledge of this relationship can be interpreted as specialized knowledge that the arbitrageur as a sophisticated investor has access to. The lack of cross-conditioning coupled with the arbitrageur's specialized knowledge about the relationship between the two assets means that (1) constitutes a

privileged position insofar that it permits the arbitrageur to take positions in each market that'll allow her to profit off of her specialized knowledge by trading what's essentially the same asset in different contexts.

The liquidation value is a normally distributed random variable $\tilde{v} \sim N[0, \tau_v^{-1}]$. All non-arbitrageur traders observe private noisy signals regarding signals regarding the normalized liquidation value $\tau_v^{-1/2} v$:

$$
\begin{aligned}
\forall a \in N_A, \ a \lhd i_a = \tau_a^{1/2} \tau_v^{1/2} v + e_a, \ where \ e_a \sim N[0, 1] \\
\forall z \in N_Z, \ z \lhd i_z = \tau_z^{1/2} \tau_v^{1/2} v + e_z, \ where \ e_z \sim N[0, 1]
\end{aligned}
\tag{2}
$$

where $\lhd$ is used to denote "observes", so $a \lhd b$ would read "a observes b". For purposes of tractability all random variables are assumed to be normally distributed and independent from one another. In alignment with the informational segmentation, the arbitrary j-trader can not distinguish between the arbitrageur and any other J-trader, and treats her as just another J-trader. This is because the traders don't believe that there exists an arbitrageur, the disbelief in an arbitrageur follows from the assumption of overconfidence that follows; the existence of an arbitrageur would present the possibility of someone having access to more precise information and contradict the assumption.

Trading will ultimately be motivated through overconfidence in signal precision, each non-arbitrageur trader believes that their signal is of high precision τ whilst the signal of all other market participants is of low precision π. Trader's agree to disagree about their beliefs regarding the relative precision of their signals, put formally:

$$
\begin{aligned}
\forall a \in N_A, \ \tau_a = \tau > \pi = \tau_{a'}, \ \forall a' \neq a, a' \in L_A \\
\forall z \in N_Z, \ \tau_z = \tau > \pi = \tau_{z'}, \ \forall z' \neq z, z' \in L_Z
\end{aligned}
\tag{3}
$$

Each trader believes themselves to be endowed with a high precision signal whilst everyone else is stuck with low precision signals, and they agree to disagree about who has this coveted high precision signal. The arbitrageur does not observe a private signal. It is this disagreement in signal precision which motivates trading in the assets which are in net zero supply:

$$
\eta_A + \sum_{a=1}^{N_A} s_a = \eta_Z + \sum_{z=1}^{N_Z} s_z = 0
\tag{4}
$$

where η_J denotes the arbitrageurs' current inventory in market J and s_j denotes the current inventory of the j-trader in that market. Each trader $a \in N_A$ and $z \in N_Z$ trades the amounts x_a and x_z respectively, and the arbitrageur trades the amount ψ_A of asset A in market A and ψ_Z of asset Z in market Z, market clearing requires that:

$$\psi_A + \sum_{a=1}^{N_A} x_a = \psi_Z + \sum_{z=1}^{N_Z} x_z = 0 \tag{5}$$

It is common knowledge that the asset is in zero-net supply, and while the traders may agree to disagree about who's signal has high precision of τ and who has a low precision π signal they do agree on the values of τ and π, additionally each trader in market J knows that the population of market participants is $|L_J| = N_J + 1$.

As previously mentioned, the non-arbitrageur traders in each market treat the arbitrageur as if she were just another overconfident trader. Someone who assigns a low precision to the information inferred by the trades of others and assigns a high precision to their own information. In effect were the arbitrageur to come clean and tell one of the other traders about the other market and their privileged position, the other trader would simply not believe them treating the information as cheap talk. Since traders agree to disagree about their beliefs concerning signal precision and can't distinguish the arbitrageur, believing the arbitrageurs story would contradict these assumptions and subsequently collapse into a no-trade result.

2.1 Linear Trade Conjectures

All non-arbitrageur traders conjecture that every other market participant trades linearly in their private signal i_j, the market price P_J, and their inventory s_j:

$$\forall a \in N_A, \ a \text{ believes that: } x_a' = \beta_A i_a' - \gamma_A P_A - \delta_A s_a' \ , \forall a' \in L_A$$
$$\forall z \in N_Z, \ z \text{ believes that: } x_z' = \beta_Z i_z' - \gamma_Z P_Z - \delta_Z s_z' \ , \forall z' \in L_Z \tag{6}$$

We proceed to solve for the trades x_j^* in the same way as in Kyle, Obizhaeva, & Wang (2017). Summing up across all the other traders the market clearing condition implies:

$$-x_j = \sum_{k \in L_J} x_k = \beta_A \sum_{k \in L_J} i_k - N_J \gamma_J P_J - \delta_J \sum_{k \in L_J \backslash j} s_k \tag{7}$$

allowing i_{-j} denote the average non-j signal which is to say $i_{-j} = \frac{1}{N_J} \sum_{k \in L_J \setminus j} i_k$, and using the zero net supply condition we get:

$$-x_j = N_J \beta_A i_{-j} - N_J \gamma_J P_J + \delta_J s_j \tag{8}$$

Rearranging [8] to write the price as a function of j's trade, x_j and inventory s_j:

$$P_J = \frac{\beta_J}{\gamma_J} i_{-j} + \frac{\delta_J}{N_J \gamma_J} s_j + \frac{1}{N_J \gamma_J} x_j \tag{9}$$

In equilibrium j knows the values of the constants, β_J, γ_J, and δ_J; since j will also know their trade x_j and inventory s_j, the price P_J will be a sufficient statistic for the average other signal i_{-j}, this means that the j-trader's information set is composed of $\{i_j, i_{-j}\}$. Let $Var^j[v]$ denote j's estimate of the variance of the liquidation value v conditioned on their information, so from j's perspective the total precision in the market is determined by:

$$\tau_J = (Var^j[v])^{-1} = \tau_v(1 + \tau + N_J \pi) \tag{10}$$

Since all the stochastic variables are jointly normally distributed, then by the projection theorem, j's expectation of the liquidation value conditioned on their information, denoted, v_j, is given by:

$$v_j = \frac{\tau_v^{1/2}}{\tau_J}(\tau^{1/2} i_j + N_J \pi^{1/2} i_{-j}) \tag{11}$$

2.2   Utilities and Objective Functions

All traders, as well as the arbitrageur, share the same form of time additively separable expected utility of $\langle -e^{-\rho w_j} \rangle$ where ρ is a common/shared CARA parameter and w_j denotes j's terminal wealth. Each A- and Z-trader choses their trade x_a or x_z in order to maximize their expected utility, whilst the arbitrageur choses both her trade in market A, ψ_A and her trade in Z ψ_Z so as to maximize her own expected utility. Trader j's terminal expected terminal wealth is given by their net position multiplied by their expectation of the liquidation value, less the cost of the change in their position. Trader j assigns an expected value of to their terminal wealth as well as it's variance, conditioned on his information of:

$$\langle w_j \rangle = v_j(x_j + s_j) - x_j P_J(x_j), \quad \text{and} \quad Var^j[w_j] = (s_j + x_j)^2 Var^j[v] = \frac{1}{\tau_J}(s_j + x_j)^2 \tag{12}$$

So j's expected utility is given by:

$$\langle -e^{-\rho w_j} \rangle = -exp\Big[-\rho(v_j(x_j + s_j) - x_j P_J(x_j)) + \frac{\rho^2}{2\tau_J}(s_j + x_j)^2 \Big] \qquad (13)$$

Maximizing the expected utility function above is equivalent to maximizing the following objective function:

$$obj_j(x_j) = v_j(x_j + s_j) - x_j P_J - \frac{\rho}{2\tau_J}(x_j + s_j)^2 \qquad (14)$$

Since the arbitrageur shares the same kind of utility function their expectations of their terminal wealth and it's variance is the same as that of the j-trader but for the fact that the arbitrageur is able to take positions in both assets A and Z:

$$\langle w_{arb} \rangle = v_{arb}(\psi_A + \eta_A + \psi_Z + \eta_Z) - \psi_A P_A - \psi_Z P_Z$$
$$Var^{arb}[w_{arb}] = \frac{1}{\tau_{arb}}(\psi_A + \eta_A + \psi_Z + \eta_Z)^2 \qquad (15)$$

Where v_{arb} is the arbitrageurs expectation of the liquidation value conditioned on her available information and τ_{arb} denotes the arbitrageurs view of total precision, both of these quantities will be explicitly defined later. Given that the arbitrageur shares the same form of CARA type utility function as the other traders, maximizing the expected utility of $\langle U(w_{arb}) \rangle = \langle -e^{-\rho w_{arb}} \rangle$ is equivalent to maximizing the following objective function:

$$obj_{arb}(\psi_A, \psi_Z) = v_{arb}(\psi_A + \eta_A + \psi_Z + \eta_Z) - \psi_A P_A - \psi_Z P_Z - \frac{\rho}{2\tau_{arb}}(\psi_A + \eta_A + \psi_Z + \eta_Z)^2$$
$$(16)$$

Definition: (Segmented Market Equilibrium with Overconfidence) We define a linear trading equilibrium as a set of prices and linear trading strategies of the A-trader, the Z-traders, and arbitrageur in market A $\{(\psi_A^*, \{x_a^*\}_{a \in N_A}, P_A^*\}$ and a set of trading strategies $\{(\psi_Z^*, \{x_z^*\}_{z \in N_Z}, P_Z^*\}$ such that each non-arbitrageur trader maximizes their objective functions as well as the arbitrageur, such that both the markets clear and the zero net supply

conditions hold:

$$\forall a \in N_A, \; x_a^* \in \arg\max_{x_a} \left[v_a(x_a + s_a) - x_a P_A - \frac{\rho}{2\tau_A}(x_a + s_a)^2 \right]$$

$$\forall z \in N_Z, \; x_z^* \in \arg\max_{x_z} \left[v_z(x_z + s_z) - x_z P_Z - \frac{\rho}{2\tau_Z}(x_z + s_z)^2 \right]$$

$$(\psi_A^*, \psi_Z^*) \in \arg\max_{(\psi_A, \psi_Z)} \left[v_{arb}(\psi_A + \eta_A + \psi_Z + \eta_Z) - \psi_A P_A - \psi_Z P_Z - \frac{\rho}{2\tau_{arb}}(\psi_A + \eta_A + \psi_Z + \eta_Z)^2 \right]$$

$$\eta_A + \sum_{a=1}^{N_A} s_a = \eta_Z + \sum_{z=1}^{N_Z} s_z = \psi_A + \sum_{a=1}^{N_A} x_a = \psi_Z + \sum_{z=1}^{N_Z} x_z = 0$$

$$(17)$$

The trading equilibrium described above is the same as the traditional notion of a Bayesian Nash Equilibrium less the presumption of a shared common prior. The lack of a common prior is needed in order to have the market participants agree to disagree (Aumann 1976) about their relative overconfidences.

2.3   Solving For Non-Arbitrageur Trades

We first solve for the optimal trades of the A- and Z-traders. Plugging equations [9] and [11] into [14] we have trader j choosing x_j so to maximize:

$$obj(x_j) = \frac{\tau_v^{1/2}(s_j + x_j)\left(\tau^{1/2}i_j + \pi^{1/2}N_J i_{-j}\right)}{\tau_J} - \frac{x_j\left(s_j\delta_J + x_j + \beta_J N_J i_{-j}\right)}{\gamma_J N_J} - \frac{\rho(s_j + x_j)^2}{2\tau_J}$$

$$(18)$$

Solving for x_j by the objective function's first order condition yields:

$$x_j = \frac{\tau^{1/2}i_j\gamma_J N_J\tau_v^{1/2} - \rho s_j\gamma_J N_J - s_j\delta_J\tau_J - \beta_J N_J\tau_J i_{-j} + \pi^{1/2}\gamma_J N_J^2 i_{-j}\tau_v^{1/2}}{\rho\gamma_J N_J + 2\tau_J} \qquad (19)$$

Rearranging equation [9] to solve for i_{-j}, yields:

$$i_{-j} \to \frac{-s_j\delta_J - x_j + \gamma_J N_J P_J}{\beta_J N_J} \qquad (20)$$

Plugging in [20] into [19] and rearranging yields:

$$x_j = \left[\frac{\tau^{1/2}\beta_J\gamma_J N_J\tau_v^{1/2}}{\beta_J\tau_J + \gamma_J N_J\left(\rho\beta_J + \pi^{1/2}\tau_v^{1/2}\right)}\right]i_j + \left[\frac{\gamma_J N_J\left(-\rho\beta_J - \pi^{1/2}\delta_J\tau_v^{1/2}\right)}{\beta_J\tau_J + \gamma_J N_J\left(\rho\beta_J + \pi^{1/2}\tau_v^{1/2}\right)}\right]s_j$$

$$+ \left[\frac{\gamma_J N_J\left(\pi^{1/2}\gamma_J N_J\tau_v^{1/2} - \beta_J\tau_J\right)}{\beta_J\tau_J + \gamma_J N_J\left(\rho\beta_J + \pi^{1/2}\tau_v^{1/2}\right)}\right]P_J$$

$$(21)$$

Clearly the j-trader trades linearly in their private signal i_j, the prevailing market price P_J, and their current inventories s_j in agreement with the linear trade conjecture. Given that the j-trader believes that the rest of the market is populated with similarly overconfident traders, they trade believing in a symmetric equilibrium meaning that we can solve for the trade coefficients β_J, γ_J, and δ_J by matching the coefficients in the linear conjecture [6] with those in equation [21]

$$
\begin{aligned}
\beta_J &= \frac{\tau_v^{1/2} \left(N_J \left(\tau^{1/2} - 2\pi^{1/2} \right) - \tau^{1/2} \right)}{\rho N_J} \\
\gamma_J &= \frac{\tau_J \left(N_J \left(\tau^{1/2} - 2\pi^{1/2} \right) - \tau^{1/2} \right)}{\rho N_J \left(\pi^{1/2} N_J + \tau^{1/2} \right)} \\
\delta_J &= \frac{N_J \left(\tau^{1/2} - 2\pi^{1/2} \right) - \tau^{1/2}}{N_J \left(\tau^{1/2} - \pi^{1/2} \right)}
\end{aligned}
\tag{22}
$$

By plugging in the beta and gamma values in [22] into equation [19] and simplifying allows us to write the trades by:

$$
\begin{aligned}
x_j &= \frac{\tau_v^{1/2} \left(i_j - i_{-j} \right) \left(N_J \left(\tau^{1/2} - 2\pi^{1/2} \right) - \tau^{1/2} \right)}{\rho \left(N_J + 1 \right)} - s_j \delta_J \\
x_j &= \frac{\beta_J N_J \left(i_j - i_{-j} \right)}{N_J + 1} - s_j \delta_J
\end{aligned}
\tag{23}
$$

And plugging [22], [23] into the price function [9] results in the price being equal to the average expected liquidation value of the traders in the market. In this set up the traders must be overconfident enough in their signal precision so that they'd place a high enough weight on their own information so that they believe it's profitable to trade in the direction of their private signal and not just in the direction of the average valuation for which the prevailing price is a sufficient statistic. The required amount of overconfidence needed is given by checking the second order condition of the trader's optimization problem.

2.4 Disagreement Requirement

A key to the validity of the trader optimization is the second order condition that would require $\frac{2}{\gamma_J N_J} + \frac{\rho}{\tau_J} > 0$ this condition can hold if and only if:

$$
\frac{\tau^{1/2}}{\pi^{1/2}} - 2 - \frac{\tau^{1/2}}{\pi^{1/2} N_J} > 0
\tag{24}
$$

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This is identical to the requirement in Kyle, Obizhaeva, Wang (2017) that their measure of disagreement denoted Δ_H is greater than zero with a trading population of $N_J + 1$ rather than N_J. Simply put each traders needs to believe that their signals are more than twice as precise as those of the other traders in order for them to put a large enough weight on their own information such that there'll be enough disagreement amongst the traders to motivate trade. This condition guarantees that $\delta_J > 0$

2.5 Implied Signal and Price Function

If from the point of view of the arbitrary j-trader, the arbitrageur trades according so some signal denoted i_{arb}^J, this is the signal that would have to be observed by a non-arbitrageur in order to engage in a trade that's identical to that of the arbitrageur. Backing out this 'implied signal' from the the average non-j-trader signal $i_{-j} = \frac{1}{N_J}\left(i_{arb}^J + \sum_{k \in N_J \setminus \{j\}} i_k\right)$ and adding up all the non-arbitrageur trades implies through market clearing and zero net supply:

$$\psi_J = -\sum_{j=1}^{N_J} x_j = -\frac{N_J \beta_J}{(N_J + 1)}\left(\sum_{j=1}^{N_J} i_j + \frac{1}{N_J}\sum_{j=1}^{N_J}\left(i_{arb}^J + \sum_{k \in N_J \setminus \{j\}} i_k\right)\right) + \delta_J \sum_{j=1}^{N_J} s_j$$

$$\psi_J = -\frac{N_J \beta_J}{(N_J + 1)}\left(N_J \bar{i}_J - i_{arb}^J - (N_J - 1)\bar{i}_J\right) - \delta_J \eta_J = \frac{N_J \beta_J}{(N_J + 1)}\left(i_{arb}^J - \bar{i}_J\right) - \delta_J \eta_J \tag{25}$$

Hence the arbitrageur trades as if she were an arbitrary j-trader who observed i_{arb}^J, given the beliefs of the non-arbitrageur traders the arbitrageur knows that she'll be mistaken for just such a trader. If the arbitrageur trades a high positive amount of the asset the market reacts as if a trader got a high positive signal $i_{arb}^J >> 0$, if she sells a high amount of the asset the market would react as if someone got a very negative signal $i_{arb}^J << 0$. This is the mechanism through which the prices react to the arbitrageurs trades. The implied signals i_{arb}^A and i_{arb}^Z are presumed to take the form of $i_{arb}^J = \pi^{1/2}\tau_v^{1/2}v + e_{arb}$, where $e_{arb} \sim N[0,1]$. By writing the price as a function of the implied signal and the average J-trader signal it is easily shown that the equilibrium price from the perspective of the J-trader is the average of all the trader valuations:

$$P_J = \frac{\tau_v^{1/2}}{\tau_J}\left(\frac{\tau^{1/2} + N_J \pi^{1/2}}{N_J + 1}\right)\left(i_{arb}^J + N_J \bar{i}_J\right) \tag{26}$$

Writing the implied arbitrageur signal as a function of the arbitrageur trade and inventory:

$$i_{arb}^J = i_J + \frac{(N_J + 1)(\delta_J \eta_J + \psi_J)}{\beta_J N_J} \tag{27}$$

And plugging equation [27] into [26] and simplifying results in the price function:

$$P_J = \left[\frac{\tau_v^{1/2}\left(\pi^{1/2}N_J + \tau^{1/2}\right)}{\tau_J}\right]i_J + \left[\frac{\delta_J \tau_v^{1/2}\left(\pi^{1/2}N_J + \tau^{1/2}\right)}{\beta_J N_J \tau_J}\right]\eta_J + \left[\frac{\tau_v^{1/2}\left(\pi^{1/2}N_J + \tau^{1/2}\right)}{\beta_J N_J \tau_J}\right]\psi_J \tag{28}$$

Which after substitution reduces to $P_J = \frac{\beta_J}{\gamma_J}i_J + \frac{\delta_J}{\gamma_J N_J}\eta_J + \frac{1}{\gamma_J N_J}\psi_J$ which is consistent with the price function from equation [9]. Allowing $u_A = \frac{\beta_A i_A}{\gamma_A}$ and $u_Z = \frac{i_Z \beta_Z}{\gamma_Z}$ where u_J is the average non-arbitrageur valuation in market J, the price function becomes:

$$P_J = u_J + \frac{\delta_J}{N_J \gamma_J}\eta_J + \frac{1}{N_J \gamma_J}\psi_J \tag{29}$$

2.6 Solving for Arbitrageur Optimal Trades

Similar to the non-arbitrageur traders, since the price is given by equation [28] the arbitrageur can, by observing the prevailing market price infer i_J the average non-arbitrageur signal in that market. The arbitrageur's privileged position allows her to observe the prevailing prices in both market A and market Z, and agree to disagree assumption would imply that the arbitrageur assigns a low precision π to the non-arbitrageur signals, so effectively the arbitrageur observes the following two "constructed" signals given that the arbitrageur assigns low precision to all the non-arbitrageur traders:

$$arb \lhd i_A = \frac{1}{N_A}\sum_{a=1}^{N_A} i_a = \frac{1}{N_A}\sum_{a=1}^{N_A}(\tau_a^{1/2}\tau_v^{1/2}v + e_a) = \pi^{1/2}\tau_v^{1/2}v + N_A^{-1}\sum_{a=1}^{N_A} e_a = \pi^{1/2}\tau_v^{1/2}v + \bar{e}_A$$

$$arb \lhd i_Z = \frac{1}{N_Z}\sum_{z=1}^{N_Z} i_z = \frac{1}{N_Z}\sum_{z=1}^{N_Z}(\tau_z^{1/2}\tau_v^{1/2}v + e_z) = \pi^{1/2}\tau_v^{1/2}v + N_Z^{-1}\sum_{z=1}^{N_Z} e_z = \pi^{1/2}\tau_v^{1/2}v + \bar{e}_Z \tag{30}$$

Where $\bar{e}_A \sim N[0, N_A^{-1}]$ and $\bar{e}_Z \sim N[0, N_Z^{-1}]$ by signal independence. $\forall a, a' \in N_A$, $a \neq a'$, $\tilde{e}_a \perp \tilde{e}_{a'}$, and $\forall z, z' \in N_Z$, $z \neq z'$, $\tilde{e}_z \perp \tilde{e}_{z'}$. Scaling by the square root of the market trader population to get the signals in the same form as [2], the arbitrageur observes the

following two scaled signals:

$$arb \lhd h_A = \sqrt{N_A}\pi^{1/2}\tau_v^{1/2}v + \sqrt{N_A}\bar{e}_A = \sqrt{N_A}\pi^{1/2}\tau_v^{1/2}v + \epsilon_A, \; where \; \epsilon_A \sim N[0,1]$$
$$arb \lhd h_Z = \sqrt{N_Z}\pi^{1/2}\tau_v^{1/2}v + \sqrt{N_Z}\bar{e}_Z = \sqrt{N_Z}\pi^{1/2}\tau_v^{1/2}v + \epsilon_Z, \; where \; \epsilon_Z \sim N[0,1] \quad (31)$$

The arbitrageur uses the information from these two inferred signals to construct an expectation of the liquidation value v_{arb} and an estimation of total precision τ_{arb}. In other words the arbitrageur effectively observes N_A A-traders receiving an average signal of i_A and N_Z Z-traders receiving an average signal of i_Z, and so assigns a value to total precision of;

$$\tau_{arb} = \tau_v \left(N_A\pi + N_Z\pi + 1\right) \quad (32)$$

and per the projection theorem, the arbitrageurs expected liquidation value is:

$$v_{arb} = \frac{\tau_v^{1/2}\left(i_A N_A \pi^{1/2} + i_Z N_Z \pi^{1/2}\right)}{\tau_{arb}} \quad (33)$$

Plugging her estimation of total precision [32], her expectation of the liquidation value [33], and the price functions [29] into [16] returns the objective function:

$$obj_{arb} \rightarrow \frac{\pi^{1/2}\tau_v^{1/2}\left(i_A N_A + i_Z N_Z\right)\left(\eta_A + \psi_A + \eta_Z + \psi_Z\right)}{\tau_{arb}} - \frac{\rho\left(\eta_A + \psi_A + \eta_Z + \psi_Z\right)^2}{2\tau_{arb}}$$
$$-\psi_A\left(\frac{\delta_A\eta_A + \psi_A}{\gamma_A N_A} + u_A\right) - \psi_Z\left(\frac{\delta_Z\eta_Z + \psi_Z}{N_Z\gamma_Z} + u_Z\right) \quad (34)$$

Solving for the optimal trades ψ_A and ψ_Z by first order condition yields after some algebra:

$$\psi_A = \left[-\frac{\gamma_A N_A\left(2u_A\tau_{arb} + \rho N_Z\gamma_Z\left(u_A - u_Z\right)\right)}{2\left(\rho\gamma_A N_A + 2\tau_{arb} + \rho N_Z\gamma_Z\right)}\right] + \left[\frac{\gamma_A N_A\tau_{arb}}{\rho\gamma_A N_A + 2\tau_{arb} + \rho N_Z\gamma_Z}\right]v_{arb}$$
$$+\left[\frac{\rho\gamma_A N_A\left(\delta_Z - 2\right)}{2\left(\rho\gamma_A N_A + 2\tau_{arb} + \rho N_Z\gamma_Z\right)}\right]\eta_Z - \left[\frac{\left(\delta_A\left(2\tau_{arb} + \rho N_Z\gamma_Z\right) + 2\rho\gamma_A N_A\right)}{2\left(\rho\gamma_A N_A + 2\tau_{arb} + \rho N_Z\gamma_Z\right)}\right]\eta_A$$
$$\psi_Z = \left[\frac{N_Z\gamma_Z\left(\rho\gamma_A N_A\left(u_A - u_Z\right) - 2\tau_{arb}u_Z\right)}{2\left(\rho\gamma_A N_A + 2\tau_{arb} + \rho N_Z\gamma_Z\right)}\right] + \left[\frac{\tau_{arb}N_Z\gamma_Z}{\rho\gamma_A N_A + 2\tau_{arb} + \rho N_Z\gamma_Z}\right]v_{arb} \quad (35)$$
$$+\left[\frac{\rho\left(\delta_A - 2\right)N_Z\gamma_Z}{2\left(\rho\gamma_A N_A + 2\tau_{arb} + \rho N_Z\gamma_Z\right)}\right]\eta_A - \left[\frac{\left(\delta_Z\left(\rho\gamma_A N_A + 2\tau_{arb}\right) + 2\rho N_Z\gamma_Z\right)}{2\left(\rho\gamma_A N_A + 2\tau_{arb} + \rho N_Z\gamma_Z\right)}\right]\eta_Z$$

The No-Information Presumption It is worth recognizing that so far we have placed no restriction on the arbitrageurs trading and hence can make use of her privileged position in the market to not only engage in the trades that take advantage of any differences in prices but also to construct their own expectation of the liquidation value based off of the

two average market signals i_A and i_Z and trade on that information. Since we're interested in the level of the trading dedicated to taking advantage of price differences we look to restrict our arbitrageurs trading. It can be shown that the introducing a restriction that requires ending inventories to be net zero, $\eta_A + \psi_A + \eta_Z + \psi_Z = 0$, the optimal trades under this restriction is equivalent to the limit of the unrestricted trades as $\tau_{arb} \to 0$. The condition that $\tau_{arb} \to 0$ can be interpreted as the arbitrageur presuming there to be no valid informative value in any of the signals (i.e. sees them as noise) and as a result the arbitrageur would not place any weight on any portion of their trades that are reliant on their expectation of the liquidation value. This can also be seen as an exogenous restriction that the arbitrageur is disallowed from holding any exposure at the end of trading. This "no information" assumption by the arbitrageur will correspond to the following optimal trades

$$\psi_A = \left[\frac{\gamma_A N_A N_Z \gamma_Z}{2\left(\gamma_A N_A + N_Z \gamma_Z\right)} \right] (u_Z - u_A) - \left[\frac{(2\gamma_A N_A + \delta_A N_Z \gamma_Z)}{2\gamma_A N_A + 2N_Z \gamma_Z} \right] \eta_A + \left[\frac{\gamma_A N_A (\delta_Z - 2)}{2\left(\gamma_A N_A + N_Z \gamma_Z\right)} \right] \eta_Z$$

$$\psi_Z = \left[\frac{\gamma_A N_A N_Z \gamma_Z}{2\left(\gamma_A N_A + N_Z \gamma_Z\right)} \right] (u_A - u_Z) + \left[\frac{(\delta_A - 2) N_Z \gamma_Z}{2\left(\gamma_A N_A + N_Z \gamma_Z\right)} \right] \eta_A - \left[\frac{(\gamma_A N_A \delta_Z + 2N_Z \gamma_Z)}{2\gamma_A N_A + 2N_Z \gamma_Z} \right] \eta_Z$$

(36)

Presuming no-information is equivalent to taking the limit of the trades without the presumption as $\tau_{arb} \to 0$. Note that $\psi_A + \psi_Z = -(\eta_A + \eta_Z)$ consistent with the condition that the arbitrageur takes perfectly offsetting positions in each market in terms of the prevailing market prices

$$\psi_A = -\frac{\gamma_A N_A \left(\eta_A + N_Z \gamma_Z \left(P_A - P_Z\right) + \eta_Z\right)}{\gamma_A N_A + N_Z \gamma_Z}$$

$$\psi_Z = \frac{N_Z \gamma_Z \left(-\eta_A + \gamma_A N_A \left(P_A - P_Z\right) - \eta_Z\right)}{\gamma_A N_A + N_Z \gamma_Z}$$

(37)

2.7 Equilibrium

The equilibrium is characterized by the set of non-arbitrageur trades, arbitrageur trades, and prices $\{(\psi_A^*, \{x_a^*\}_{a \in N_A}, P_A^*), (\psi_Z^*, \{x_z^*\}_{z \in N_Z}, P_Z^*)\}$ such that each trade maximizes their individual objective functions and markets clear:

$$\forall j \in N_J \;:\; x_j^* = \frac{\beta_J N_J \left(i_j - i_{-j}\right)}{N_J + 1} - s_j \delta_J \;\; \forall J \in \{A, Z\}$$

$$\psi_A^* = \left[\frac{\gamma_A N_A N_Z \gamma_Z}{2\left(\gamma_A N_A + N_Z \gamma_Z\right)} \right] (u_Z - u_A) - \left[\frac{(2\gamma_A N_A + \delta_A N_Z \gamma_Z)}{2\gamma_A N_A + 2N_Z \gamma_Z} \right] \eta_A + \left[\frac{\gamma_A N_A (\delta_Z - 2)}{2\left(\gamma_A N_A + N_Z \gamma_Z\right)} \right] \eta_Z$$

$$\psi_Z^* = \left[\frac{\gamma_A N_A N_Z \gamma_Z}{2(\gamma_A N_A + N_Z \gamma_Z)}\right](u_A - u_Z) + \left[\frac{(\delta_A - 2)N_Z \gamma_Z}{2(\gamma_A N_A + N_Z \gamma_Z)}\right]\eta_A - \left[\frac{(\gamma_A N_A \delta_Z + 2N_Z \gamma_Z)}{2\gamma_A N_A + 2N_Z \gamma_Z}\right]\eta_Z$$

Defining the optimal inferred signal i_{arb}^{J*} in market J by evaluating equation [27] at the arbitrageur trade ψ_J^* in [36]. Then given the set of signals $\{i_1, i_2, ..., i_{N_J}\}$ in market J, the equilibrium market price in that market will be the average valuation using the implied signal i_{arb}^{J*} as well:

$$P_J^* = \frac{1}{N_J + 1}\left(v_J(i_{arb}^{J*}) + \sum_{j=1}^{N_J} v_j(i_j)\right) = \frac{\tau_v^{1/2}(\tau^{1/2} + N_J \pi^{1/2}}{\tau_J}\left(\frac{1}{N_J + 1}i_{arb}^{J*} + \frac{N_J}{N_J + 1}\bar{i}_J\right)$$

$$\text{where: } i_{arb}^{J*} = i_J + \frac{(N_J + 1)(\delta_J \eta_J + \psi_J^*)}{\beta_J N_J}$$

2.8 Trading Speed

Following Kyle, Obizhaeva, Wang (2017) we define the j-trader's target inventory, ϕ_j as the level of inventory at which the trader would choose not to trade, given by solving for s_j such that $x_j^* = 0$.:

$$\phi_j = \frac{\beta_J N_J (i_j - i_{-j})}{\delta_J(N_J + 1)} \tag{38}$$

Doing this such allows us to rewrite the equilibrium trades as:

$$x_j^* = \delta_J(\phi_j - s_j) \tag{39}$$

Hence δ_J can be interpreted as the "trading speed", or rate at which the j-traders adjust their current inventories towards their target levels. Clearly $\delta_J \in (0, 1)$, since $\delta_J > 1$ would mean that the j-trader is overshooting his target inventory level whilst $\delta_J < 0$ would imply that they're trading away from their targets. $\delta_J = 1$ corresponds to the perfect competition case since the invariance of the price to individual trades allows a trader to trade all the way to their targets immediately; and $\delta_J = 0$ corresponds to the trivial no-trade result. Approaching the arbitrageurs trading in a similar fashion her target inventories in A and Z are:

$$\phi_A = -\frac{\gamma_A N_A (\tau_{arb}\delta_Z (u_A - v_{arb}) + \rho N_Z \gamma_Z (u_A - u_Z))}{\delta_Z(\delta_A \tau_{arb} + \rho\gamma_A N_A) + \rho\delta_A N_Z \gamma_Z}$$
$$\phi_Z = \frac{N_Z \gamma_Z (\delta_A \tau_{\text{arb}}(v_{arb} - u_Z) + \rho\gamma_A N_A (u_A - u_Z))}{\delta_Z(\delta_A \tau_{arb} + \rho\gamma_A N_A) + \rho\delta_A N_Z \gamma_Z} \tag{40}$$

Restricting our attention to the no-information case ($\tau_{arb} \to 0$) simplifies the target inventories to:

$$\phi_A = -\frac{\gamma_A n_A n_Z \gamma_Z \left(u_A - u_Z\right)}{\delta_A n_Z \gamma_Z + \gamma_A n_A \delta_Z} = \phi_{arb} \ \ \text{and} \ \ \phi_Z = \frac{\gamma_A n_A n_Z \gamma_Z \left(u_A - u_Z\right)}{\delta_A n_Z \gamma_Z + \gamma_A n_A \delta_Z} = -\phi_{arb} \qquad (41)$$

Note that these target inventories are consistent with the no-information presumption's implication of zero net exposure in that $\phi_A + \phi_Z = 0$, which allows us to write $\phi_A = \phi_{arb}$ and $\phi_Z = -\phi_{arb}$. We can think of ϕ_{arb} as a measure of target depth of investment into the convergence trade by the arbitrageur. In the arbitrageur's case ϕ_{arb} can be loosely interpreted as how deep of an investment into the convergence trade they'd like to make if their trading did not effect the prevailing prices. When $\phi_{arb} = 0$ we can interpret it as the arbitrageur as wanting to close out of their positions in the markets, something that'll happen whenever $u_A = u_Z$ so there's no difference in the average non-arbitrageur valuations in either market. This makes sense since without any disagreement between the markets, engaging in the convergence trade would return no profits and the arbitrageur would want to correct any existing imbalances in inventories so to net out her exposure. A variable of interest is how this depth of target investment changes with the equilibrium trading speeds in each market:

$$\partial_{\delta_A}\phi_{\text{arb}} = \frac{N_A N_Z^2 \left(u_A - u_Z\right) \gamma_A \gamma_Z^2}{\left(N_Z \gamma_Z \delta_A + N_A \gamma_A \delta_Z\right)^2}, \ \partial_{\delta_Z}\phi_{\text{arb}} = \frac{N_A^2 N_Z \left(u_A - u_Z\right) \gamma_A^2 \gamma_Z}{\left(N_Z \gamma_Z \delta_A + N_A \gamma_A \delta_Z\right)^2} \qquad (42)$$

Presuming without any loss in generality that $u_A < u_Z$ the level of target investment decreases in the trading speed in each market. Rewriting the arbitrageur's optimal trades in the similar form $\psi_J^* = d_J(\phi_J - \eta_J)$ and solving for d_J returns the relatively messy equations:

$$d_A = \frac{\left(\delta_A N_Z \gamma_Z + \gamma_A N_A \delta_Z\right)\left(\gamma_A N_A \left(2\eta_A + N_Z \gamma_Z \left(u_A - u_Z\right) - \left(\delta_Z - 2\right)\eta_Z\right) + \delta_A \eta_A N_Z \gamma_Z\right)}{2\left(\gamma_A N_A + N_Z \gamma_Z\right)\left(\gamma_A N_A \left(N_Z \gamma_Z \left(u_A - u_Z\right) + \eta_A \delta_Z\right) + \delta_A \eta_A N_Z \gamma_Z\right)}$$

$$d_Z = \frac{\left(\delta_A N_Z \gamma_Z + \gamma_A N_A \delta_Z\right)\left(\gamma_A N_A \left(N_Z \gamma_Z \left(u_A - u_Z\right) - \delta_Z \eta_Z\right) + N_Z \gamma_Z \left(\left(\delta_A - 2\right)\eta_A - 2\eta_Z\right)\right)}{2\left(\gamma_A N_A + N_Z \gamma_Z\right)\left(\gamma_A N_A \left(N_Z \gamma_Z \left(u_A - u_Z\right) - \delta_Z \eta_Z\right) - \delta_A N_Z \gamma_Z \eta_Z\right)}$$

$$(43)$$

The important thing to keep in mind about d_A and d_Z is that neither is equal to δ_A or δ_Z.

2.9   Price Differences

The law of one price would imply that $P_Z - P_A = 0$ so naturally we'd be interested if this price difference increases or decreases with the arbitrageur's trading, and how the arbitrageur

would have to trade in order to cause the prices to converge. We first consider the case when the arbitrageur does not trade in either market nor holds any inventory so in effect this case corresponds to when there's no arbitrageur.

$$\text{Price Difference Without Arbitrageur: } P_Z^0 - P_A^0 = u_Z - u_A \tag{44}$$

An interesting result is that the price difference when incorporating the optimal arbitrageur trades is invariant to whether or not its restricted by the no-information assumption or not.

$$P_Z^* - P_A^* = \frac{1}{2}(u_Z - u_A) + \frac{1}{2}\left(\frac{\delta_Z}{N_Z \gamma_Z}\eta_Z - \frac{\delta_A}{\gamma_A N_A}\eta_A\right) \tag{45}$$

When the arbitrageur begins with no initial inventory in either market $\eta_A = \eta_Z = 0$ then the price difference is halved by the arbitrageur's trades and the arbitrageur's trading does indeed help to bring the prices closer to the law of one price. As for the question of how the arbitrageur would have to trade in order to cause the prices to converge it's convenient to view the trades in the form $\psi_J = d_J(\phi_J - \eta_J)$. When the arbitrageur adjusts their inventories towards their targets at their optimal rates d_A and d_Z, the price difference is [45] however were the arbitrageur to trade at the prevailing trading speed in each market instead, so $\psi_A = \delta_A(\phi_{arb} - \eta_A)$ and $\psi_Z = \delta_Z(-\phi_{arb} - \eta_Z)$ the resulting price difference would be 0. So if the arbitrageur trades towards her target inventories in each market at the same rate at which the non-arbitrageur traders do, there will be no price gap, so its the fact that they do not trade towards their real target inventories at the prevailing market speeds that keeps the prices from converging due to the arbitrage trade. This is the result of the arbitrageur exercising her monopolist power over her privileged position. Though the price difference does crucially rely on the arbitrageur's initial inventories namely if $\frac{\delta_Z}{N_Z \gamma_Z}\eta_Z - \frac{\delta_A}{N_A \gamma_A}\eta_A \geq u_Z - u_A$ then the prices would diverge rather than converge. Without any loss of generality let's presume that $u_Z > u_A$ so that intuitively the arbitrageur trades would cause a price divergence if her position in the "overpriced" asset Z is sufficiently more long than her preexisting position in the "underpriced" asset A. The degree to which the initial inventories have to differ is driven by the prevailing trading speeds in each market. In this one-period model we take the initial inventories as exogenously given, in order to examine how an arbitrageur would even find themselves in such a position endogenously a

multi-period model is needed, something we explore next, and then in a simple two period setup in the proceeding sections.

3 General Multiperiod Model

In this generalization of the single period model structure, at the beginning of each period the traders observe a noisy signal about the fundamental value and trade on that signal knowing that there's some probability that the liquidation value will be revealed at the end of trading otherwise it'll be revealed later and they start a new round of trading with new signals. We assume a T-finite set of liquidation values $\{v_1, v_2, ..., v_T\}$ where each is identically independently normally distributed, $v_t \sim N[0, \tau_v^{-1}]$. At the beginning of each trading round each non-arbitraguer trader observes a noisy signal of the form $j \lhd i_{j,t} = \tau_j^{1/2} \tau_v^{1/2} v_t + e_{j,t}$, where $e_{j,t} \sim N[0, 1]$; the error terms in the signals are i.i.d. across the traders as well as across time. The assumption that the varying liquidation values are independently distributed is a simplifying assumption which may not need be the case, in a possible extension in which the liquidation values are correlated the correlation coefficients between v_t and all $v_{t=n}, n > 0$ would determine the rate of signal decay. Viewed in this light the independence of liquidation values means that we're treating the informativeness of the signals as short lived and decay immediately.

After a liquidation value is revealed the game ends, for each trading round t there exists some probability p_t that the liquidation value v_t will be revealed at the end of round t and the game ends after round t. Conditional on reaching round t the probability of the game ending after that round is given by q_t (so with probability of $(1 - q_t)$ the game continues). So $p_t = q_t \prod_{k=1}^{t-1}(1 - q_k)$. Having $q_T = 1$ means that the game will end by round T with certainty since the probability of continuing to round T+1 is zero. These probabilities are common knowledge. The finites turn-based nature of the generalized model means that it can be solved through backward induction.

Imagine you found yourself at the start of round T and it's common knowledge the liquidation value v_T will be revealed at the end of trading, clearly the decision problem being

faced here is identical to the one in the single period model. Since we've already solved the single period model, we already know the optimal trading strategies employed at time T. For the arbitrary j-trader define the potential objective function at time t as:

$$o_{j,t} = v_{j,t}(x_{j,t} + s_{j,t}) - x_{j,t}P_{J,t} - \sum_{n=1}^{t-1} x_{j,n}P_{J,n} - \frac{\rho}{2\tau_J}(x_{j,t} + s_{j,t})^2 \qquad (46)$$

Where the period t inventory $s_{j,t}$ is the consequence of their initial inventory and history of trades $s_{j,t} = s_{j,1} + \sum_{n=1}^{t-1} x_{j,n}$. At time T the j-trader's objective function is equal to $o_{j,T}$, rolling back one period to T-1 the j-trader knows that there's a probability q_{T-1} that they'll receive a payoff according to $o_{j,T}$ and of $(1-q_{T-1})$ that v_{T-l} will not be revealed and will thus implement the known optimal single period trading strategy next turn. Not knowing what the period T signals will be, the j-trader instead takes the expected payoff implementing that trading strategy, so $obj_{j,T-1} = (q_{T-1})o_{j,T-1} + (1 - q_{j,T-1})\langle o_{j,T} \rangle$. More generally the j-trader trades $x_{j,t}$ at time t in order to maximize:

$$obj_{j,t} = (q_t)o_{j,t} + (1 - q_{j,t})\langle obj_{j,t+1} \rangle \qquad (47)$$

As in the single period model the linear trade conjecture is made by the traders, so they believe that in every turn traders trade according to $x_{j,t} = \beta_{J,t}i_{j,t} - \gamma_{J,t}P_{J,t} - \delta_{J,t}s_{j,t}$, $\forall t \in \{1, 2, ..., T\}$. Since at the last period T the game reduces to the single period model the optimal j-traders trades correspond to that in [23] and the coefficients [22].

3.1   Trading Speed Invariance

The transition from a static single period model to that of a dynamic multiperiod model introduces added complexity to the equilibrium trading strategies. For some intuition consider the case of $\gamma_{J,t}$, recall that $\gamma_{J,t}$ denotes the sensitivity of $x_{j,t}$ to the prevailing price in that market at that time $P_{J,t}$. At time T it's known with certainty that $P_{J,T} \to v_T$ with certainty, whereas at time t there's a q_t chance that $P_{J,t} \to v_t$ otherwise $P_{J,t} \to P_{J,t+1}$ and $P_{J,t+1}$ is unknown at time t when the decision on $x_{j,t}$ needs to be made, so it naturally makes sense that $\gamma_{J,t} \neq \gamma_{J,t'}$. A similar argument follows with the signal coefficient $\beta_{J,t}$, which results in the term $u_{J,t} = \frac{\beta_{J,t}i_{J,t}}{\gamma_{J,t}}$ reflecting a dampened average of the valuations, i.e. the weights on the

individual valuations sum up to a quantity less than one, this is similar to the price dampening described in Kyle, Obizheava, Wang (2017) though they described it with a continuous trading model. In their continuous trading model they limit their problem to linear trading solutions where the trading coefficients are presumed to be constant; interestingly in our multiperiod generalization that assumption is not made and the trading trading coefficients do vary, with the exception of δ_J. In other words it turns out that $\delta_{J,1} = \delta_{J,2} = ... = \delta_{J,T}$.

Derivation of Trading Speed Invariance: We begin with the invariance result which is obtained by evaluating the final round's objective function with the optimal trades then taking its expectation and plugging it into equation [47]. Doing the same at time T-1 we get to the T-1 objective function, and solve for the coefficients by matching with the optimal trades for time T-1, T-2, and so on. One finds that the trade coefficient for private signals $\beta_{J,T-n}$ is given by:

$$\beta_{J,T-n} = \frac{q_{T-n}}{\mu_{J,T-n}}\beta_{J,T} \tag{48}$$

And the trading speed can always be written as:

$$\delta_{J,T-n} = \frac{\mu_{J,T-n}\rho(N_J - 1)\beta_{J,T-n}}{N_J(\rho\beta_{J,T-n} + q_{T-n}\pi^{1/2}\tau_v^{1/2}) + q_{T-n}\pi^{1/2}\tau_v^{1/2}} \tag{49}$$

Substituting [48] into [49] the q_{T-n}'s and $\mu_{J,T-n}$'s cancel out and $\delta_{J,T-n}$ becomes constant:

$$\delta_{J,T-n} = \frac{\rho(N_J + 1)\beta_{J,T}}{N_J(\rho\beta_{J,T} + \pi^{1/2}\tau_v^{1/2}) + \pi^{1/2}\tau_v^{1/2}} = \frac{N_J\left(\tau^{1/2} - 2\pi^{1/2}\right) - \tau^{1/2}}{N_J\left(\tau^{1/2} - \pi^{1/2}\right)} = \delta_J \tag{50}$$

Where the variable $\mu_{J,T-n}$ evolves according to the backward recursion formula:

$$\mu_{J,T-n} = 1 + (1 - q_{T-n})(\delta_J - 2)\delta_J(\mu_{J,T-n+1} + (1 - q_{T-n})) \tag{51}$$

And the price trade coefficient:

$$\gamma_{J,T-n} = \frac{\tau_J}{q_{T-n}(\tau^{1/2} + N_J\pi^{1/2})}\beta_{J,T-n}$$
$$n \in \{0, 1, ..., T - 1\} \tag{52}$$

Outline of δ Invariance Proof. Intuitively we solve the last objective function at time T to get the time T coefficients, we then move from the time T objective to the time T-1

objective according [47] and solve for the T-1 coefficients. If we presume that $x_{j,T-n-1} =$

$\beta_{J,T-n-1}i_{j,T-n-1}-\gamma_{J,T-n-1}P_{J,T-n-1}-\delta_{J,T}s_{j,T-n-1} = \frac{\beta_{J,T-n-1}N_J(i_{j,T-n-1}-i_{-j,T-n-1})}{N_J+1}-s_{j,T-n-1}\delta_{J,T}$

where $s_{j,T-n-1} = s_{J,T-n} + x_{j,T-n}$ maximizes $obj_{j,T-n-1}$, ommiting the algebra, evaluating

$obj_{j,T-n} = (q_{T-n})o_{j,T-n} + (1 - q_{T-n})\langle obj_{j,T-n-1}\rangle$ at $x_{j,T-n-1}$ and simplifying yields:

$$obj_{j,T-n} = \frac{x_{j,T-n}}{2N_J\gamma_{J,T-n}\tau_J}\Big[- 2(x_{j,T-n} + \delta_{J,T-n}s_{j,T-n})\tau_J$$

$$+N_J((-1 - 2(-1 + q_{T-n})\delta_{J,T} + (-1 + q_{T-n})\delta_{J,T}^2)(2\rho s_{j,T-n}\gamma_{J,T-n} + \rho x_{j,T-n}\gamma_{J,T-n})$$

$$-2i_{-j,T-n}\beta_{J,T-n}\tau_J + 2q_{T-n}\tau^{1/2}\tau_v^{1/2}\gamma_{J,T-n}i_{j,T-n}) + 2q_{T-n}\pi^{1/2}\tau_v^{1/2}N_J^2\gamma_{J,T-n}i_{-j,T-n}\Big]$$

$$+g$$

(53)

Where g is not effected by $x_{j,T-n}$ and thus has no bearing on the choice of $x_{j,T-n}$. Solving by FOC will yield $x_{j,T-n}^*$ and by plugging in $-\partial_{P_{J,T-n}}[x_{j,T-n}^*]$ for $\gamma_{J,T-n}$ (by matching with the LTC) into the delta matching equation $\delta_{J,T-n} = -\partial_{s_{j,T-n}}[x_{j,T-n}^*]$ returns:

$$\delta_{J,T-n} = \frac{\rho(N_J - 1)\beta_{J,T-n}(-1 - 2(-1 + q_{T-n})\delta_{J,T} + (-1 + q_{T-n})\delta_{J,T}^2)}{N_J(\rho\beta_{J,T-n}(-1 - 2(-1 + q_{T-n})\delta_{J,T} + (-1 + q_{T-n})\delta_{J,T}^2) - q_{T-n}\pi^{1/2}\tau_v^{1/2}) - q_{T-n}\pi^{1/2}\tau_v^{1/2}}$$

(54)

Which corresponds to equation [49] which reduces to [50]. Since this result holds for $n = 1$ it holds for all values $n \in \{1, ..., T - 1\}$ via induction.

Since $\delta_T = \delta_{T-1}$ it would make sense that if δ turned out to be invariant to the set of operations we executed to get from the time T objective to the time T-1 coefficients then it should also be invariant when we apply the exact same set of operations to get from the T-1 objective to the T-2 coefficients and indeed when manually working out the algebra that's what is found. The attempt to give structure to the evolution of coefficients from one period to the next results in the equations [48], [49], [50], [51], [52].

4   2-Period Example

So far the arbitrageurs optimal trades in every case thus examined are in the direction of price correction, she would never willingly make trade in the direction of price divergence. Even in the multiperiod model if $u_{A,T-1} < u_{Z,T-1}$ she may trade to acquire a positive position in A and a negative position in Z only to have $u_{A,T} > u_{Z,T}$ in this case her inventory effects

could serve to cause a price divergence though she would trade aggressively to reverse her positions in A and Z, and the divergent prices serve to help subsidize her aggressive trading. Depending on the sensitivities of each market (faster markets a less sensitive and hence allow more aggressive trading) she can decide how much, if at all to shade her trades across time.

For illustrative purposes we consider a simple, specialized case of the multiperiod model restricted to two periods a short term $t = 1$ and a long term $t = 2$ with a perfectly competitive market in the initially overpriced asset Z: $u_{Z,1} > u_{A,1}$, in other words $N_Z \to \infty$ so $\delta_Z \to 1$ and the arbitrageur's trades and inventory effects are nonexistent. Meanwhile the market in the initially underpriced asset A is less than perfectly competitive $N_A < \infty$, so $\delta_A \in (0, 1)$. The case that $N_J \to \infty$ implies that $\delta_J \to 1$ is the result of Kyle Obizhaeva, & Wang's (2017) second theorem.

The arbitrageur begins with no initial inventory in either asset $\eta_{Z,1} = \eta_{A,1} = 0$ and takes on a short position in asset Z, $\psi_{Z,1} = -\psi_1$, that is offset by a long position in A, $\psi_{A,1} = \psi_1$. There's a probability $p \in (0, 1)$ that the prices will converge in the short term which will happen when the liquidation value v_1 is revealed. If the prices do not converge in the short term then they'll converge in the long term, if however the prices in market Z rise further before the converge takes place then the arbitrageur will incur "temporary" losses, these temporary losses effect the arbitrageur subsequent trades through the simple constraint requiring the arbitrageur to cover any temporary losses through trading in A.

$$(P_{Z,2} - P_{Z,1})(-\psi_1) - \psi_{A,2} P_{A,2} \geq 0 \tag{55}$$

It should be noted that the no-information assumption is crucial here since the price in the market with an infinite population is certain to converge to the true value and so the arbitrageur would simply trade in that direction rather than deal with the offsetting positions required in a convergence trade. So the prevailing market price in the Z market would seem to completely random to the arbitrageur.

If the price in Z rises in the second period then the constraint would bind and $\psi_{A,2} = \frac{(P_{Z,2} - P_{Z,1})}{P_{A,2}}(-\psi_1)$, since the price in Z is symmetrically distributed centrally at zero the probability of the constraint binding is 1/2. Were the constraints not bind the optimal

trades would be:

$$\psi_{A2} = \frac{1}{2}\left(n_A\gamma_{A2}\left(P_{Z2} - u_{A2}\right) - \psi_1\delta_{A2}\right)$$

$$\psi_{Z2} = \frac{1}{2}\left(n_A\gamma_{A2}\left(u_{A2} - P_{Z2}\right) + \psi_1\delta_{A2}\right)$$

(56)

If the constraint binds the optimal trades would be:

$$\psi_{A2} = \frac{1}{2}\left(n_A\gamma_{A2}\sqrt{\frac{4\psi_1\left(P_{Z1} - P_{Z2}\right)}{n_A\gamma_{A2}} + \left(\frac{\psi_1\delta_{A2}}{n_A\gamma_{A2}} + u_{A2}\right)^2} - n_A\gamma_{A2}u_{A2} - \psi_1\delta_{A2}\right)$$

(57)

$$\psi_{Z,2} = -\psi_{A,2}$$

The object of interest is the price gap, so we look at hoe it is effected by changes in the market speed in A, δ_A. Taking derivatives of $(P_{Z,2} - P_{A,2})$ evaluated in the constrained case:

$$\partial_{\delta_A}\left(P_{Z2} - P_{A2}\right) = -\frac{1}{2}n_A\psi_1\gamma_{A2}\left(\frac{u_{A2}}{\sqrt{4n_A\psi_1\gamma_{A2}\left(P_{Z1} - u_{Z2}\right) + u_{A2}^2}} + 1\right)$$

(58)

and in the unconstrained case:

$$\partial_{\delta_A}\left(P_{Z2} - P_{A2}\right) = -n_A\psi_1\gamma_{A2}$$

(59)

Note that the two derivatives are negative, since $\delta_Z = 1$ and $\delta_A \in (0,1)$, a rise in the trading speed in A corresponds to a drop in the difference in trading speeds, in this context $\delta_A \uparrow \Leftrightarrow \delta_Z - \delta_A \downarrow$. When $\delta_Z - \delta_A \uparrow$ the price difference $P_{Z,2} - P_{A,2} \uparrow$ and since $\frac{4(P_{Z,1} - P_{Z,2})}{N_A\gamma_{A,2}}\psi_{A,1} < 0$

$$\partial_{\delta_A}(P_{Z,2} - P_{A,2})|_{constrained} < \partial_{\delta_A}(P_{Z,2} - P_{A,2})|_{unconstrained}$$

5   Possible Empirical Tests

Going forward one hopes to empirically test the comparative statics implied by the model namely that the wider the gap in trading speed between two markets, the lower will be any arbitrage activity between the two. There are two major issues in taking the model to the empirical side, the first and most obvious is the question of how one measures trading speed. If I want to hold 100 shares of xyz and go out and by 20 shares a day for the next 5 days my δ is equal to 0.2, but note that those trades would be indistinguishable from the case of someone wanting to hold 200 shares of xyz and by 20 shares a day for the next 10 days, i.e. $\delta = 0.1$. Since as a researcher all thats available is the trade data

and not the target inventories it does not seem clear how one would estimate trading speed from the data. In order to deal with this I would propose using the methodology proposed Kyle & Obizhaeva's (2015,2016,2017) recent work concerning their empirical hypothesis of market microstructure invariance and the application of dimensional analysis. If their market microstructure invariance hypothesis are assumed then deep lying model characteristics like the trade of speed are to be proportional to certain transformations of readily accessible trade data such as volatility, volume, and price. When viewed with one eye on this kind of analysis the time invariance of trading speed in our model is a great result as it satisfies their invariance hypothesis.

Having a presumably acceptable empirical methodology for estimating trading speed from trading data we now need instances where we can test the model's implication that mispricings will be higher the greater the difference in trading speeds. This presents a whole new mess of issues, because one would need to find instances which could be considered mispricings. For a clean test one would find a set of clear mispricings like those surveyed in Lamont & Thaler (2003). Using their methodology on equity carve-outs one could first presumably expand their sample by using more current instances as well and obtain trading data on the carve-out stock and the stock of the parent company in the time surrounding the carve-out and measure to what extent the differences in the trading speeds in the two markets explain the variation in the mispricings.

6 Conclusion

Considering the textbook case of a mispricing where the same asset is being traded at different prices in different markets. The no arbitrage argument follows that arbitrageurs will buy in the lower priced market, causing the price to rise, and sell in the higher priced market, causing the price to fall, the two effects together serve to diminish the mispricing. In reality we still observe instances of mispricings and are fascinated by them. There are broadly two ways we try to explain these anomalies, the first is to show that wasn't an anomaly at all, though they may look similar if A is exposed to some risk factor that Z is

not exposed to then the difference in price could just be reflective of that fact, so there's no need for the arbitrageur, and so when a mispricing arises it may continue to persist if the constraints on the arbitrageur are limiting enough. The spirit of this paper runs somewhere in between the two approaches instead of saying asset A isn't really the dame as asset Z we say that differences between the markets discourages arbitrage and in cases where the arbitrageur is constrained these differences add more bite to the constraints.

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